THE BRIDGES OF KÖNIGSBERG - A HISTORICAL PERSPECTIVE

by

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Abstract

The Bridges of Königsberg is one of the most famous problems in graph theory. In the summer of 2005, two of the authors visited Königsberg, now called Kaliningrad. This article provides geographical and historical information on Königsberg and its bridges, as well as updated information on the current day situation.

Key-words: graph theory, arc routing, history, Euler.

1. Introduction

The Bridges of Königsberg is one the most famous problems in graph theory and is a standard feature of textbooks in the area of arc routing. In the early 18th century, the inhabitants of Königsberg, now called Kaliningrad, debated whether there existed a closed walk that crossed exactly once each of the seven bridges of the river Pregel. The problem was solved by the Swiss mathematician Leonhard Euler (1707-1783), who was at that time a chair of mathematics at the St. Petersburg Academy of Sciences. He proved that no such walk could exist and reported his findings in Euler (1736). This article contains a drawing, which is reproduced in Newman (1953) and to which we have added geographical designations (Figure 1).
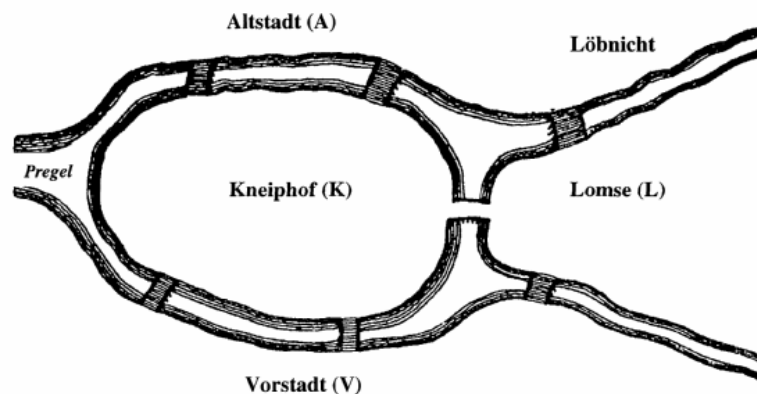

Figure 1. Euler's drawing of the bridges of Königsberg in 1736

To solve the problem, Euler drew an undirected graph (Figure 2) in which the four vertices represent Altstadt and Löbnicht (A) located on the north shore of the Pregel, Vorstadt (V) located on the south shore, and the two islands of Kneiphof (K) and Lomse (L). Each of the seven edges corresponds to a bridge. Euler argued that for a solution to exist in an undirected graph such as this one, the graph

must be connected, and each of its vertices must have an even degree, which is clearly not the case here. A graph containing a closed walk using each edge exactly once is now called *unicursal* or *Eulerian*.
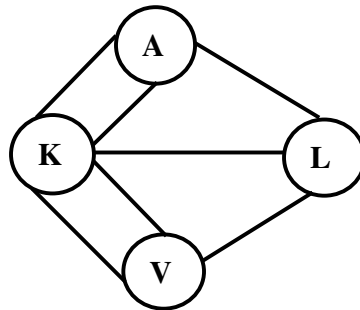

Figure 2. Graphical representation of the bridges of Königsberg in 1736

Interestingly, Euler did not prove that connectedness and evenness are sufficient conditions for unicursality. It was Hierholzer (1873) who provided the first polynomial-time "end-pairing" algorithm for the determination of a unicursal walk in a connected and even undirected graph. A simple description of this algorithm can be found in Edmonds and Johnson (1973), while several alternative procedures are described in Fleischner (1991).

The necessary and sufficient conditions for unicursality have since been extended to directed and mixed graphs (Ford and Fulkerson, 1962). Hierholzer's end-pairing algorithm can easily be adapted to Eulerian directed graphs. However, most graphs encountered in practice are not Eulerian. A common problem is then to determine a least cost traversal of all edges of the graph, as in the *Chinese Postman Problem* (Guan, 1962), or of a subset of them as in the *Rural Postman Problem* (Orloff, 1974). Several extensions and variants of these two basic problems have also been proposed, giving rise to a rich research area. Surveys of the most common arc routing problems are provided by Eiselt, Gendreau and Laporte (1995a, 1995b), Assad and Golden (1995), and Eiselt and Laporte (2000).

An English translation of Euler's original article was published in *Scientific American* (Newman, 1953), and some historical information can be found in Biggs, Lloyd and Wilson (1976). While the problem of the Bridges of Königsberg is well known, most references contain little or no background information on the city and its bridges. In the summer of 2005 two of the authors visited Kaliningrad and gathered some geographical and historical facts which we believe will be of interest to the readers of *Networks*, particularly researchers and teachers in the area of arc routing. Our two main bibliographical sources of information are The New Encyclopædia Britannica (1994) and Bartfeld (2005).

2. Königsberg

Königsberg has been called Kaliningrad since 1946. The modern city is the capital of the Kaliningrad province (oblast), a Russian exclave wedged between Poland and Lithuania (Figure 3), which was once a part of East Prussia. It is named after Mikhail Ivanovich Kalinin (1875-1946) who was for many years the formal head of government in the Soviet Union. Kaliningrad is located very close to the Baltic Sea, only a couple of kilometres upstream along the river Pregel (Pregolya).

Figure 3. The Kaliningrad province (reprinted from http://europa.eu.int/)

There are two islands in the Pregel in the centre of Königsberg. The Island of Kneiphof is depicted and mentioned by name in most books describing the Königsberg Bridges problem. It is a rather small almost rectangular island, measuring approximately 400 metres in the east-west direction by 200 metres in the other direction. The second island, called Lomse, is always only partly depicted and unidentified in standard references. As can be seen from Figure 4, it is much larger than Kneiphof. It is approximately nine kilometres long, and its breadth varies from 200 metres to almost one kilometre. It is crossed by a canal five kilometres to the east of its western tip. Most of the island is marshland and only its westernmost part is built up.
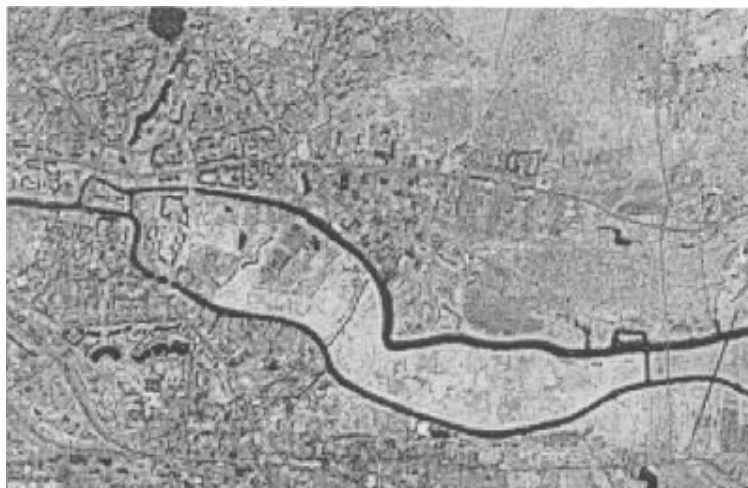

Figure 4. Aerial map of Kaliningrad showing the island of Kneiphof (top left) and the island of Lomse up to the canal (reprinted from http://commons.wikimedia.org/wiki/Kaliningrad)

The history of Königsberg goes back a long time and is rather complicated. The city, which just celebrated its 750th anniversary, was formally founded in 1255, when a castle was built on a hill on

the north bank of the Pregel, and a town grew under its protection. The town was gradually surrounded by a wall with several gates and became known as Altstadt. It obtained a city status in 1286. To the east of Altstadt, outside the wall but still on the north bank, a second city emerged because Altstadt could no longer accommodate its expanding population. This new town, called Löbnicht, was also fortified and received its city status in 1300.

The Island of Kneiphof, located just south of Altstadt, was ideal as a port for the ships trading between the Baltic cities and the towns of Altstadt and Löbnicht. Kneiphof soon became an important market-place and grew into an autonomous city in 1327. Thus, from this year on, three small cities were located very close to each other on the island Kneiphof and on the north bank of the Pregel. These were ruled independently of each other with separate aldermen and city councils. In 1340 all three became members of the Hanseatic league. The south bank was called Vorstadt, but very few people lived there. The three cities remained separated for almost another 400 years. Altstadt was recognised for its military power, Kneiphof for its wealth, and Löbnicht for its mud. In 1724 the three cities merged into one and were given the name of Königsberg – the Royal Mountain.

The northeastern part of Kneiphof was the site of the University of Königsberg which was founded by Albrecht of Prussia (1490-1568) in 1544, under the name of Albertus University or Albertina. It is one of the oldest universities in Europe. In 1862, the university was relocated to the centre of what was once Altstadt. It has hosted many famous researchers. The best known is probably the philosopher Immanuel Kant (1724-1804) who spent whole his life in Königsberg. He is buried in an open-air mausoleum behind the north wall of the Königsberg Cathedral on Kneiphof. The University of Königsberg was also the home of several well-known scientists, including the astronomer Friedrich Wilhelm Bessel, the physicists Robert Gustav Kirchoff, Franz Ernst Neumann and Arnold Sommerfeldt, and the mathematicians Alfred Clebsch, Ludwig Otto Hesse, David Hilbert, Karl Gustav Jakob Jacobi, and Hermann Minkowski. While Euler is firmly associated with Königsberg, there is no evidence that he ever visited the city. During our visit to Kaliningrad we were unable to find any statue, plaque or inscription related to him.

3. The bridges of Königsberg

We now provide a historical account of the bridges of the central part of Königsberg until the last one was built in 1905. These are identified in Figure 5 in the order of their construction date.

The Salesman's bridge [1]

The first bridge of Königsberg dating from 1286 connected Altstadt and Kneiphof. It belonged to the city of Altstadt and was erected to provide easy access to the market place on Kneiphof. Originally named the St. George's bridge, it became later known as the Krämerbrücke – the Salesman's bridge, but in Hanseatic times it was also called Koggenbrücke, after the cog, the most common ship used by the Hanseatic league. The bridge could be lifted to provide access further up the river for sailing vessels. It was reconstructed in 1787 and was again rebuilt in steel in 1900.

The Green bridge [2]

A second bridge, completed in 1322, connected Kneiphof to the south bank and provided easy access to Vorstadt and to the hinterland. The bridge was called Grünebrücke – the Green bridge, because of the green waves in the background of the coat of arms of Kneiphof. It was destroyed by

fire in 1582 and rebuilt in 1590, still in wood. It remained in this state until 1907 when it was reconstructed in steel.
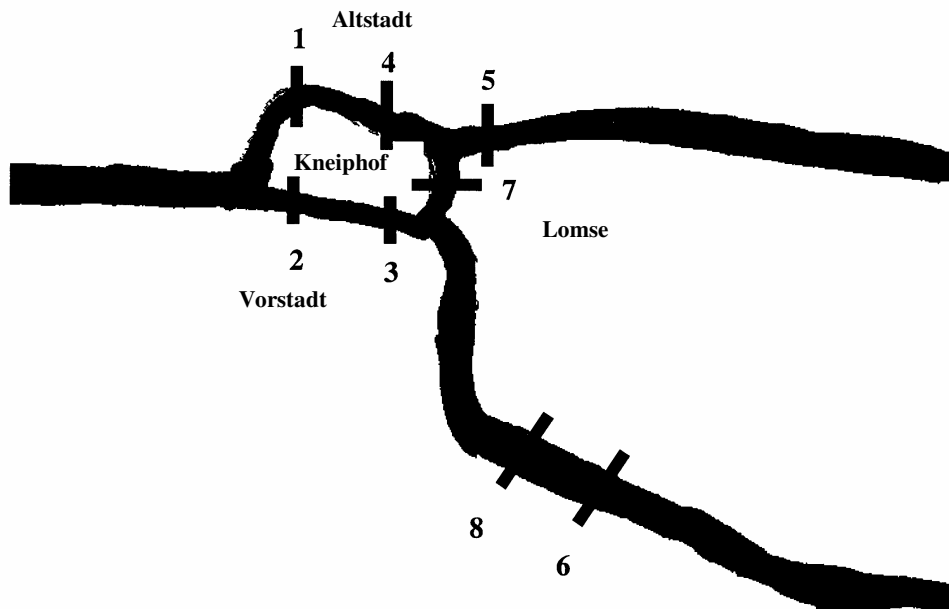

Figure 5. The bridges of Königsberg in 1905

The Slaughter bridge [**3**]

In the 14th century, a slaughterhouse operated on the south bank of the Pregel. In order to facilitate the transportation of meat to the three cities, a new bridge called Köttelbrücke – the Slaughter bridge, was constructed in 1377 between the south bank and Kneiphof. The Slaughter bridge was rebuilt in steel in 1886.

The Blacksmith's bridge [**4**]

Between 1333 and 1380 a cathedral was built in the northeastern part of Kneiphof. It was isolated from the rest of the island by a wall. A pedestrian bridge, called Dombrücke – the Church bridge, was therefore erected to allow the deans of the cathedral to connect their lands on Altstadt to the cathedral. The Church bridge was demolished in 1379 and a new bridge, called Schmiedebrücke – the Blacksmith's bridge, was built slightly to the west of it in 1397. The Blacksmith's bridge started from a part of Altstadt where many blacksmiths had their shops. If a graph like that of Figure 2 had been drawn at that time, it would have been Eulerian. This would, however, have been the case for only a very short period since a new bridge was soon to be added between Altstadt and Lomse and the graph never became Eulerian again. The Blacksmith's bridge was rebuilt in 1787 in wood, and in 1896 in steel.

The Timber bridge [5]

The Island of Lomse was used to stock timber and a new bridge completed some time between 1400 and 1404 made it easier to transport the wood to the two north bank cities. The bridge was named Holzbrücke – the Timber bridge. It was maintained for 500 years and in 1904 it was rebuilt in steel next to its old location.

The High bridge [6]

The next bridge, called Hohebrücke – the High bridge, was constructed in 1506 in order to connect Lomse to the south bank. It was reconstructed in 1882-1883, dismantled in 1937 and rebuilt a few meters to the east in steel and concrete in 1937-1939. The foundations of the original bridge can still be seen today.

The Honey bridge [7]

The seventh of Euler's bridges, connecting the two islands, was completed in 1542. It was constructed by the inhabitants of Kneiphof who wanted to access the north bank without having to pass over the two bridges from Kneiphof controlled by Altstadt. According to the legend, the town council of Kneiphof gave a large barrel of honey to the mayor of Altstadt in order to obtain permission to construct the bridge. The bridge was therefore called Hönigbrücke – the Honey bridge. It was rebuilt in steel between 1879 and 1882. So, in 1542 all seven bridges of Köningsberg considered by Euler were in place.

The Emperor's bridge [8]

No more bridges were constructed in the city centre of Königsberg until 1905, when a new bridge connecting Lomse with the expanding city of Vorstadt on the south bank was erected. The bridge was given the name Kaiserbrücke – the Emperor's bridge, in honour of the emperor Wilhelm I of Germany (1797-1888).

4. The bridges of Kaliningrad

After World War II, Königsberg became part of the Soviet Union and acquired its new name. The islands of Kneiphof and Lomse later became known as Kant Island and Oktyabrskiy (October) Island. During the war four of the eight bridges of Königsberg were damaged, namely the Blacksmith's bridge, the Slaughter bridge, the Timber bridge and the Emperor's bridge. The Blacksmith's bridge and the Slaughter bridge were pulled down. The Timber bridge was repaired and its central part was replaced by a similar part from the Slaughter bridge. In the middle of the 1980s the Emperor's bridge was finally demolished but was rebuilt at the same place in 2005. In 1972 the Salesman's bridge and the Green bridge were demolished and replaced with a new bridge called the Estacada, identified by [9] in Figure 6. It goes directly over the westernmost part of Kant Island. In order to provide pedestrian access to Kant Island, two staircases lead from this bridge to the island, but all vehicular traffic to Kant Island now has to go via October Island. Hence, of the seven original Euler bridges, only the Timber bridge [5], the High bridge [6] and the Honey bridge [7] remain in their 1736 locations or close to them. The Timber bridge is now called the October

bridge by the inhabitants of Kaliningrad because it is part of October street which connects the northern and southern parts of the city through October Island.

There are still seven bridges in the centre of Kaliningrad if one interprets the two staircases of the Estacada as independent bridges connecting the river banks to Kant Island. To complete the picture, we mention the existence of two more bridges in the vicinity of Kaliningrad. Downstream, a kilometre or so to the west of Kant Island, now stands a railway and car bridge over the Pregolya, and there is also a car bridge five kilometres east of Kant Island. We have decided not to consider these two bridges as part of our account since they are not located in the city centre. A recent map of Kaliningrad shows another yet uncompleted bridge [10] (represented by the dashed line in Figure 6), but we sensed that its construction had somehow been abandoned. However, even if this project saw the light of day, with or without staircases to October Island, the associated graph would still not be Eulerian.
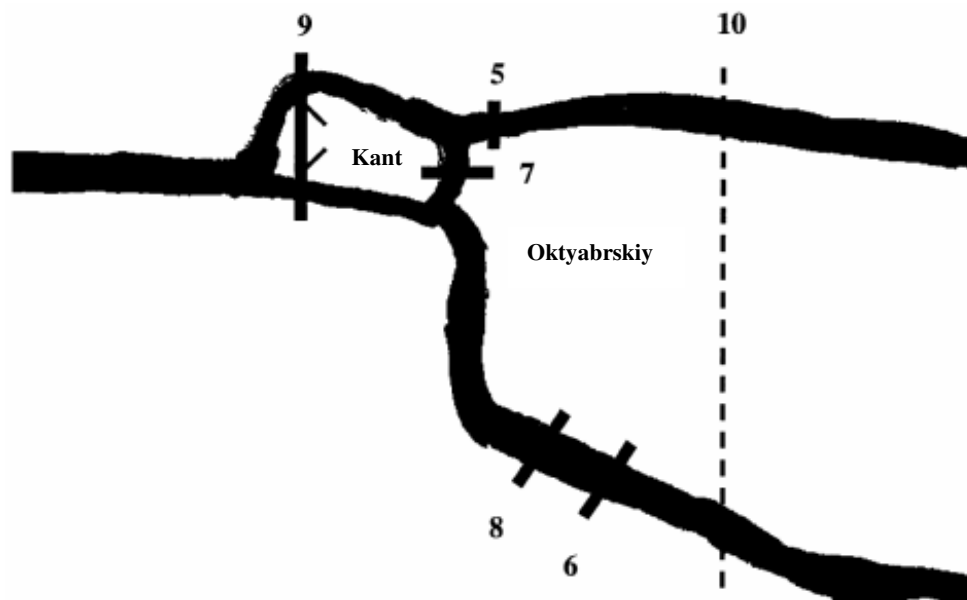

Figure 6. The bridges of Kaliningrad in 2006 with the current names of the islands

Acknowledgement

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